Chapter 4

Existence and uniqueness

4.1 Introduction

A major theoretical question in the study of ordinary differential equations is: When do solutions exist? In this chapter, we study this question, and also the question of uniqueness of solutions.

To begin with, note that in Chapter 1, we have shown the existence and uniqueness of the solution to

$$x' = Ax, \quad x(t_0) = x_0.$$

Indeed, the unique solution is given by $x(t) = e^{(t-t_0)A}x_0$.

It is too much to expect that one can show existence by actually giving a formula for the solution in the general case:

$$x' = f(x,t), \quad x(0) = x_0.$$

Instead one can prove a theorem that asserts the existence of a unique solution if the function f is not 'too bad'. We will prove the following "baby version" of such a result.

Theorem 4.1.1 *Consider the differential equation*

$$x' = f(x,t), \quad x(t_0) = C, \tag{4.1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ *is such that there exists a constant* L *such that for all* $x_1, x_2 \in \mathbb{R}$, *and all* $t \geq t_0$,

$$|f(x_1,t) - f(x_2,t)| \leq L|x_1 - x_2|.$$

Then there exists a $t_1 > t_0$ *and a* $x \in C^1[t_0,t_1]$ *such that* $x(t_0) = x_0$ *and* $x'(t) = f(x(t),t)$ *for all* $t \in [t_0,t_1]$.

We will state a more general version of the above theorem later on (which is for *nonscalar* f, that is, for a *system* of equations). The proof of this more general theorem is very similar to the proof of Theorem 4.1.1.

The next few sections of this chapter will be spent in proving Theorem 4.1.1. Before we get down to gory detail, though, we should discuss the method of proof.

Let us start with existence. The simplest way to prove that an equation has a solution is to write down a solution. Unfortunately, this seems impossible in our case. So we try a variation.

We write down functions which, while not solutions, are very good approximations to solutions: they miss solving the differential equations by less and less. Then we try to take the limit of these approximations and show that it is an actual solution.

Since all those words may not be much help, let's try an example. Suppose that we want to solve $x^2 - 2 = 0$. Here's a method: pick a number $x_0 > 0$, and let

$$\begin{aligned} x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right), \\ x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right), \\ x_3 &= \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right), \end{aligned}$$

and so on. We get a sequence of numbers whose squares get close to 2 rather rapidly. For instance, if $x_0 = 1$, then the sequence goes

$$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{656657}{470832}, \dots,$$

and the squares are

$$1, 2\frac{1}{4}, 2\frac{1}{144}, 2\frac{1}{166464}, 2\frac{1}{221682772224}, \dots.$$

Presumably the x_n's approach a limit, and this limit is $\sqrt{2}$. Proving this has two parts. First, let's see that the numbers approach a limit. Notice that

$$x_n^2 - 2 = \frac{1}{4}\left(x_{n-1} + \frac{2}{x_{n-1}}\right)^2 - 2 = \frac{1}{4}\left(x_{n-1} - \frac{2}{x_{n-1}}\right)^2 \geq 0,$$

and so $x_n^2 \geq 2$ for all $n \geq 1$. Clearly $x_n > 0$ for all n (recall that $x_0 > 0$). But then

$$x_n - x_{n-1} = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right) - x_{n-1} = \frac{1}{2x_{n-1}}(2 - x_{n-1}^2) < 0.$$

Hence $(x_n)_{n\geq 1}$ is decreasing, but is also bounded below (by 0), and therefore it has a limit.

Now we need to show that the limit is $\sqrt{2}$. This is easy. Suppose the limit is L. Then as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),$$

taking[1] limits, we get $L = \frac{1}{2}\left(L + \frac{2}{L}\right)$, and so $L^2 = 2$. So L does indeed equal $\sqrt{2}$, that is, we have constructed a solution to the equation $x^2 - 2 = 0$.

The iterative rule $x_{n+1} = (x_n + 2/x_n)/2$ may seem mysterious, and in part, it is constructed by running the last part of the proof backwards. Suppose $x^2 - 2 = 0$. Then $x = 2/x$, or $2x = x + 2/x$, or $x = (x + 2/x)/2$. Now think of this equation not as an equation, but as a formula for producing a sequence, and we are done.

Analogously, in order to prove Theorem 4.1.1, we will proceed in 3 steps:

1. Take the equation, manipulate it cleverly, and turn it into a rule for producing a sequence of approximate solutions to the equation.
2. Show that the sequence converges.
3. Show that the limit solves the equation.

[1] To be precise, this is justified (using the algebra of limits) provided that $\lim\limits_{n\to\infty} x_n \neq 0$. But since $x_n^2 > 2$, $x_n > 1$ for all n, so that surely $L \geq 1$.

4.2 Analytic preliminaries

In order to prove Theorem 4.1.1, we need to develop a few facts about calculus in the vector space $C[a,b]$. In particular, we need to know something about when two vectors from $C[a,b]$ (which are really two functions!) are "close". This is needed, since only then can we talk about a a *convergent* sequence of *approximate* solutions, and carry out the plan mentioned in the previous section.

It turns out that just as Chapter 1, in order to prove convergence of the sequence of matrices, we used the matrix norm given by (1.16), we introduce the following "norm" in the vector space $C[a,b]$: it is simply the function $\|\cdot\| : C[a,b] \to \mathbb{R}$ defined by

$$\|f\| = \sup_{t\in[a,b]} |f(t)|,$$

for $f \in C[a,b]$. (By the Extreme Value Theorem, the "sup" above can be replaced by "max", since f is continuous on $[a,b]$.) With the help of the above norm, we can discuss distances in $C[a,b]$. We think of $\|f-g\|$ as the distance between functions f and g in $C[a,b]$. So we can also talk about convergence:

Definition. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C[a,b]$. The series $\displaystyle\sum_{k=1}^{\infty} f_k$ is said to be *convergent* if there exists a $f \in C[a,b]$ such that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$, there holds that

$$\left\| \sum_{k=1}^{n} f_k - f \right\| < \epsilon.$$

The series $\displaystyle\sum_{n=1}^{\infty} f_n$ is said to be *absolutely convergent* if the real series $\displaystyle\sum_{k=1}^{\infty} \|f_k\|$ converges, that is, there exists a $S \in \mathbb{R}$ so that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$, there holds that

$$\left| \sum_{k=1}^{n} \|f_k\| - S \right| < \epsilon.$$

Now we prove the following remarkable result, we will will use later in the next section to prove our existence theorem about differential equations.

Theorem 4.2.1 *Absolutely convergent series in $C[a,b]$ converge, that is, if $\displaystyle\sum_{k=1}^{\infty} \|f_k\|$ converges in $\mathbb{R}$, then $\displaystyle\sum_{k=1}^{\infty} f_k$ converges in $C[a,b]$.*

Note the above theorem gives a lot for very little. Just by having convergence of a real series, we get a much richer converge–namely that of a sequence of functions in $\|\cdot\|$–which in particular gives pointwise convergence in $[a,b]$, that is we get convergence of an infinite family of sequences! Indeed this remarkable proof works since it is based on the notion of "uniform" convergence of the functions–convergence in the norm gives a uniform rate of convergence at all points t, which is stronger than simply saying that at each t the sequence of partial sums converge.

Proof Let $t \in [a,b]$. Then $|f_k(t)| \le \|f_k\|$. So by the Comparison Test, it follows that the real series $\sum_{k=1}^{\infty} |f_k(t)|$ converges, and so the series $\sum_{k=1}^{\infty} f_k(t)$ also converges, and let $f(t) = \sum_{k=1}^{\infty} f_k(t)$. So we obtain a function $t \mapsto f(t)$ from $[a,b]$ to $\mathbb{R}$. We will show that f is continuous on $[a,b]$ and that $\sum_{k=1}^{\infty} f_k$ converges to f in $[a,b]$.

The real content of the theorem is to prove that f is a continuous function on $[a,b]$. First let us see what we have to prove. To prove that f is continuous on $[a,b]$, we must show that it is continuous at each point $c \in [a,b]$. To prove that f is continuous at c, we must show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $t \in [a,b]$ satisfies $|t-c| < \delta$, then $|f(t) - f(c)| < \epsilon$. We prove this by finding a continuous function near f. Assume that c and ϵ have been picked. Since $\sum_{k=1}^{\infty} \|f_k\|$ is finite, we can choose an $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} \|f_k\| < \frac{\epsilon}{3}$. Let $s_N(t) = \sum_{k=1}^{N} f_k(t)$. Then s_N is continuous, since it is the sum of finitely many continuous functions, and

$$|s_N(t) - f(t)| = \left| \sum_{k=N+1}^{\infty} f_k(t) \right| \le \sum_{k=N+1}^{\infty} |f_k(t)| \le \sum_{k=N+1}^{\infty} \|f_k\| < \frac{\epsilon}{3},$$

regardless of t. (This last part is the crux of the proof.) Since s_N is continuous, we can pick $\delta > 0$ so that if $|t-c| < \delta$, then $|s_N(t) - s_N(c)| < \frac{\epsilon}{3}$. This is the delta we want: if $|t-c| < \delta$, then

$$|f(t) - f(c)| \le |f(t) - s_N(t)| + |s_N(t) - s_N(c)| + |s_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This proves the continuity of f.

The rest of the proof is straightforward. We have:

$$\left| \sum_{k=1}^{n} f_k(t) - f(t) \right| = \left| \sum_{k=n+1}^{\infty} f_k(t) \right| \le \sum_{k=n+1}^{\infty} |f_k(t)| \le \sum_{k=n+1}^{\infty} \|f_k\| \overset{n\to\infty}{\longrightarrow} 0.$$

This shows that $\sum_{k=1}^{\infty} f_k$ converges to f in $C[a,b]$. ■

Theorem 4.2.2 *Suppose* $\sum_{k=1}^{\infty} f_k$ *converges absolutely to* f *in* $C[a,b]$. *Then*

$$\sum_{k=1}^{\infty} \int_a^b f_k(t)dt = \int_a^b f(t)dt.$$

Proof We need to show that for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$\left| \sum_{k=1}^{n} \int_a^b f_k(t)dt - \int_a^b f(t)dt \right| < \epsilon.$$

Choose N such that if $n \ge N$, then $\sum_{k=n+1}^{\infty} \|f_k\| < \frac{\epsilon}{b-a}$. So

$$\left| \sum_{k=1}^{n} f_k(t) - f(t) \right| = \left| \sum_{k=n+1}^{\infty} f_k(t) \right| \le \sum_{k=n+1}^{\infty} |f_k(t)| \le \sum_{k=n+1}^{\infty} \|f_k\| < \frac{\epsilon}{b-a}.$$

But then

$$\left|\sum_{k=1}^{n}\int_a^b f_k(t)dt - \int_a^b f(t)dt\right| = \left|\int_a^b (\sum_{k=1}^{n} f_k(t) - f(t))dt\right| \leq \int_a^b |\sum_{k=1}^{n} f_k(t) - f(t)|dt \leq \int_a^b \frac{\epsilon}{b-a}dt = \epsilon.$$

■

4.3 Proof of Theorem 4.1.1

4.3.1 Existence

STEP 1. The first step in the proof is to change the equation into one which we can use for creating a sequence. The principle for doing this is a useful and important one. We want to arrange matters so that successive terms are close together. For this, integration is much better than differentiation. Two functions that are close together have their integrals close together[2], but their derivatives can be far apart[3].

So we should change (4.1) into something involving integrals. The easiest way to do this is to integrate both sides. Taking into account the initial condition, we see that we get

$$x(t) - C = \int_{t_0}^{t} f(x(t), t)dt,$$

that is,

$$x(t) = \int_{t_0}^{t} f(x(t), t)dt + C.$$

Now we can construct our sequence.

We begin with

$$x_0(t) = C,$$

and define $x_1, x_2, x_3, \ldots$ inductively:

$$\begin{aligned} x_1(t) &= \int_{t_0}^{t} f(x_0(t), t)dt + C, \\ x_2(t) &= \int_{t_0}^{t} f(x_1(t), t)dt + C, \\ &\vdots \\ x_{k+1}(t) &= \int_{t_0}^{t} f(x_k(t), t)dt + C, \end{aligned}$$

and so on. All the functions $x_0, x_1, \ldots$ are continuous functions. This is the sequence we will work with.

STEP 2. We want to show that the sequence of functions $x_0, x_1, x_2, \ldots$ converges. This sequence is the sequence of partial sums of the series

$$x_0 + (x_1 - x_0) + (x_2 - x_1) + \ldots.$$

[2] At least for a while. As the interval of integration gets large, the functions drift apart.

[3] For instance, let $f(x) = \frac{1}{10^{10}}\sin(10^{100}x)$, a function close to 0. If we integrate, then we get $-\frac{1}{10^{110}}\cos(10^{100}x)$, which is really tiny; if we differentiate, we get $10^{90}\cos(10^{100}x)$, which can get very large.

So we will prove that this series, namely

$$x_0 + \sum_{k=0}^{\infty}(x_{k+1} - x_k)$$

converges. In order to do this, we show that it converges absolutely, and then by Theorem 4.2.1, we would be done.

Thus we need to look at $x_{k+1} - x_k$:

$$x_{k+1}(t) - x_k(t) = \int_{t_0}^{t}[f(x_k(t),t) - f(x_{k-1}(t),t)]dt.$$

Since we know that

$$|f(x_k(t),t) - f(x_{k-1}(t),t)| \leq L|x_k(t) - x_{k-1}(t)|,$$

where L is a number not depending on k or t. Then

$$\begin{aligned}
|x_{k+1}(t) - x_k(t)| &\leq \int_{t_0}^{t}|f(x_{k-1}(t),t) - f(x_{k-1}(t),t)|dt \\
&\leq \int_{t_0}^{t} L|x_k(t) - x_{k-1}(t)|dt \\
&\leq \int_{t_0}^{t} L\|x_k - x_{k-1}\|dt \\
&= L(t - t_0)\|x_k - x_{k-1}\|.
\end{aligned}$$

So if we work in the interval $t_0 \leq t \leq t_0 + \frac{1}{2L}$, we have

$$\|x_{k+1} - x_k\| \leq L\frac{1}{2L}\|x_k - x_{k-1}\| = \frac{1}{2}\|x_k - x_{k-1}\|.$$

Then, as one may check using induction,

$$\|x_{k+1} - x_k\| \leq \frac{1}{2^k}\|x_1 - x_0\| \tag{4.2}$$

for all k. Thus

$$\|x_0\| + \sum_{k=0}^{\infty}\|x_{k+1} - x_k\| \leq \|x_0\| + \sum_{k=0}^{\infty}\frac{1}{2^k}\|x_1 - x_0\| < \infty,$$

and the series $x_0 + \sum_{k=0}^{\infty}(x_{k+1} - x_k)$ converges absolutely. By Theorem 4.2.1, it converges and has a limit, which we denote by x.

Step 3. Now we need to know that x satisfies (4.1). We begin by taking limits in

$$x_{k+1}(t) = \int_{t_0}^{t} f(x_k(t),t)dt + C.$$

By Theorem 4.2.2, taking limits inside the integral can be justified (see the Exercise on page 70 below), and so we get

$$x(t) = \int_{t_0}^{t} f(x(t),t)dt + C. \tag{4.3}$$

We see that $x(t_0) = C$ because the integral from t_0 to t_0 is 0. Also by the Fundamental Theorem of Calculus, x can be differentiated (since it is given as an integral), and

$$x'(t) = f(x(t),t).$$

This proves the existence.

4.3.2 Uniqueness

Finally, we prove uniqueness. Let x_1 and x_2 be two solutions. Then x_1 and x_2 satisfy the "integrated" equation (4.3) as well.

Let $t_* = \max\{t \in [0,T] \mid x_1(\tau) = x_2(\tau) \text{ for all } \tau \leq t\}$. In other words, t_* is the smallest time instant after which x_1 and x_2 start to be different. See Figure 4.1.

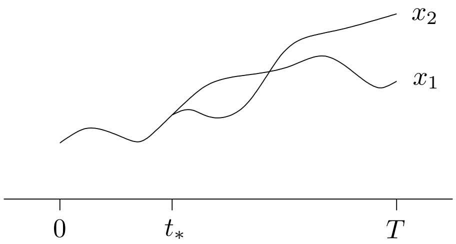

Figure 4.1: Definition of t_*.

We have

$$\begin{aligned} x_1(t) - x_1(t_*) &= \int_{t_*}^t x_1'(\tau)d\tau = \int_{t_*}^t f(x_1(\tau),\tau)d\tau, \\ x_2(t) - x_2(t_*) &= \int_{t_*}^t x_2'(\tau)d\tau = \int_{t_*}^t f(x_2(\tau),\tau)d\tau, \end{aligned}$$

and so

$$x_1(t) - x_2(t) = \int_{t_*}^t f(x_1(\tau),\tau) - f(x_2(\tau),\tau)d\tau.$$

Let $N \in \mathbb{N}$ be such that

$$N > \max\left\{1, \frac{1}{L}, \frac{1}{L(T-t_*)}\right\},$$

and let

$$M := \max_{\tau \in \left[t_*, t_* + \frac{1}{LN}\right]} |x_1(\tau) - x_2(\tau)|.$$

(Since $N > \frac{1}{L(T-t_*)}$, we know that $t_* + \frac{1}{LN} < T$.) Then for all $t \in \left[t_*, t_* + \frac{1}{LN}\right]$, we have

$$\begin{aligned} |x_1(t) - x_2(t)| &= \left| \int_{t_*}^t f(x_1(\tau),\tau) - f(x_2(\tau),\tau)d\tau \right| \\ &\leq \int_{t_*}^t |f(x_1(\tau),\tau) - f(x_2(\tau),\tau)|d\tau \\ &\leq \int_{t_*}^t L|x_1(\tau) - x_2(\tau)|d\tau \\ &\leq \int_{t_*}^t LM d\tau = LM(t - t_*) \\ &\leq LM\left(t_* + \frac{1}{LN} - t_*\right) = \frac{M}{N}. \end{aligned}$$

Thus on the interval $M \leq \frac{M}{N}$, that is, $N \leq 1$, which a contradiction to our choice of N (which satisfied $N > 1$).

This completes the proof of the theorem.

Exercise. ($*$) Justify taking limits inside the integral in STEP 3 of the proof.

4.4 The general case. Lipschitz condition.

We begin by introducing the class of Lipschitz functions.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called *locally Lipschitz* if for every $r > 0$ there exists a constant L such that for all $x, y \in B(0, r)$,

$$\|f(x) - f(y)\| \le L\|x - y\|. \tag{4.4}$$

If there exists a constant L such that (4.4) holds for all $x, y \in \mathbb{R}^n$, then f is said to be *globally Lipschitz.*

For $f : \mathbb{R} \to \mathbb{R}$, we observe that the following implications hold:

$$f \text{ is continuously differentiable} \;\Rightarrow\; f \text{ is locally Lipschitz} \;\Rightarrow\; f \text{ is continuous.}$$

That the inclusions of these three classes are strict can be seen by observing that $f(x) = |x|$ is globally Lipschitz, but not differentiable at 0, and the function $f(x) = \sqrt{x}$ is continuous, but not locally Lipschitz. (See Exercise 1c below.)

Just like Theorem 4.1.1, the following theorem can be proved, which gives a sufficient condition for the unique existence of a solution to the initial value problem of an ODE.

Theorem 4.4.1 *If there exists an $r > 0$ and a constant L such that the function f satisfies*

$$\|f(x, t) - f(y, t)\| \le L\|x - y\| \text{ for all } x, y \in B(0, r) \text{ and all } t \ge t_0, \tag{4.5}$$

then there exists a $t_1 > t_0$ such that the differential equation

$$x'(t) = f(x(t), t), \quad x(t_0) = x_0 \in B(0, r),$$

has a unique solution for all $t \in [t_0, t_1]$.

If the condition (4.5) holds, then f is said to be *locally Lipschitz in x uniformly with respect to t.*

The existence theorem above is of a local character, in the sense that the existence of a solution x_* is guaranteed only in a small interval $[t_0, t_1]$. We could, of course, take this solution and examine the new initial value problem

$$x'(t) = f(x(t), t), \quad t \ge t_1, \quad x(t_1) = x_*(t_1).$$

The existence theorem then guarantees a solution in a further small neighbourhood, so that the solution can be "extended". The process can then be repeated. However, it might happen that the lengths of the intervals get smaller and smaller, so that we cannot really say that such an extension will yield a solution for all times $t \ge 0$. We illustrate this by considering the following example.

Example. Consider the initial value problem

$$x' = 1 + x^2, \quad t \geq 0, \quad x(0) = 0.$$

Then it can be shown that $f(x) = 1 + x^2$ is locally Lipschitz in x (trivially uniformly in t). So a unique solution exists in a small time interval. In fact, we can explicitly solve the above equation and find out that the solution is given by

$$x(t) = \tan t, \quad t \in [0, \frac{\pi}{2}).$$

The solution cannot be extended to an interval larger than $[0, \frac{\pi}{2})$. ♢

Exercises.

1. (a) Prove that every continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.
   HINT: Mean value theorem.
   (b) Prove that every locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
   (c) (∗) Show that $f(x) = \sqrt{|x|}$ is not locally Lipschitz.
2. (∗) Find a Lipschitz constant for the following functions
   (a) $\sin x$ on $\mathbb{R}$.
   (b) $\frac{1}{1+x^2}$ on $\mathbb{R}$.
   (c) $e^{-|x|}$ on $\mathbb{R}$.
   (d) $\arctan(x)$ on $(-\pi, \pi)$.
3. (∗) Show that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality

$$|f(x) - f(y)| \leq L|x - y|^2 \text{ for all } x, y \in \mathbb{R},$$

   then show that f is continuously differentiable on $\mathbb{R}$.

4.5 Existence of solutions

Here is an example of a differential equation with more than one solution for a given initial condition.

Example. *An equation with multiple solutions.* Consider the equation

$$x' = 3x^{\frac{2}{3}}$$

with the initial condition $x(0) = 0$. Two of its solutions are $x(t) \equiv 0$ and $x(t) = t^3$. ♢

In light of this example, one can hope that there may be also theorems applying to more general situations than Theorem 4.4.1, which state that solutions exist (and say nothing about uniqueness). And there are. The basic one is:

Theorem 4.5.1 *Consider the differential equation*

$$x' = f(x,t),$$

with initial condition $x(t_0) = x_0$. *If the function* f *is continuous (but not necessarily Lipschitz in* x *uniformly in* t*), then there exists a* $t_1 > t_0$ *and a* $x \in C^1[t_0, t_1]$ *such that* $x(t_0) = x_0$ *and* $x'(t) = f(x(t), t)$ *for all* $t \in [t_0, t_1]$.

The above theorem says that the continuity of f is sufficient for the local *existence* of solutions. However, it does not guarantee the *uniqueness* of the solution. We will not give a proof of this theorem.

4.6 Continuous dependence on initial conditions

In this section, we will consider the following question: If we change the initial condition, then how does the solution change? The initial condition is often a measured quantity (for instance the estimated initial population of a species of fish in a lake at a certain starting time), and we would like our differential equation to be such that the solution varies 'continuously' as the initial condition changes. Otherwise we cannot be sure if the solution we have obtained is close to the real situation at hand (since we might have incurred some measurement error in the initial condition). We prove the following:

Theorem 4.6.1 *Let* f *be globally Lipschitz in* x *(with constant* L*) uniformly in* t. *Let* x_1, x_2 *be solutions to the equation* $x' = f(x,t)$, *for* $t \in [t_0, t_1]$, *with initial conditions* $x_{0,1}$ *and* $x_{0,2}$, *respectively. Then for all* $t \in [t_0, t_1]$,

$$\|x_1(t) - x_2(t)\| \leq e^{L(t-t_0)}\|x_{0,1} - x_{0,2}\|.$$

Proof Let $f(t) := \|x_1(t) - x_2(t)\|^2$, $t \in [t_0, t_1]$. If $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$, we have

$$\begin{aligned} f'(t) &= 2\langle x_1'(t) - x_2'(t), x_1(t) - x_2(t)\rangle = 2\langle f(x_1,t) - f(x_2,t), x_1(t) - x_2(t)\rangle \\ &\leq 2\|f(x_1,t) - f(x_2,t)\|\|x_1(t) - x_2(t)\| \quad \text{(by the Cauchy-Schwarz inequality)} \\ &\leq 2L\|x_1(t) - x_2(t)\|^2 = 2Lf(t). \end{aligned}$$

In other words,

$$\frac{d}{dt}(e^{-2Lt}f(t)) = e^{-2Lt}f'(t) - 2Le^{-2Lt}f(t) = e^{-2Lt}(f'(t) - 2Lf(t)) \leq 0.$$

Integrating from t_0 to $t \in [t_0, t_1]$ yields

$$e^{-2Lt}f(t) - e^{-2Lt_0}f(t_0) = \int_{t_0}^{t} \frac{d}{d\tau}e^{-2L\tau}f(\tau)d\tau \leq 0,$$

that is, $f(t) \leq e^{2L(t-t_0)}f(0)$. Taking square roots, we obtain $\|x_1(t) - x_2(t)\| \leq e^{L(t-t_0)}\|x_{0,1} - x_{0,2}\|$. ∎

Exercises.

1. Prove the Cauchy-Schwarz inequality: if $x, y \in \mathbb{R}^n$, then $|\langle x, y\rangle| \le \|x\|\|y\|$.

Cauchy-Schwarz

Hint: If $\alpha \in \mathbb{R}$, and $x, y \in \mathbb{R}^n$, then we have $0 \le \langle x+\alpha y, x+\alpha y\rangle = \langle x, x\rangle + 2\alpha\langle x, y\rangle + \alpha^2\langle y, y\rangle$, and so it follows that the discriminant of this quadratic expression is ≤ 0, which gives the desired inequality.

2. $(*)$ This exercise could have come earlier, but it's really meant as practice for the next one. We know that the solution to

$$x'(t) = x(t), \quad x(0) = a,$$

is $x(t) = e^t a$. Now suppose that we knew about differential equations, but not about exponentials. We thus know that the above equation has a unique solution, but we are hampered in our ability to solve them by the fact that we have never come across the function e^t! So we declare E to be the function defined as the unique solution to

$$E'(t) = E(t), \quad E(0) = 1.$$

   (a) Let $\tau \in \mathbb{R}$. Show that $t \mapsto E(t+\tau)$ solves the same differential equation as E (but with a different initial condition). Show from this that for all $t \in \mathbb{R}$, $E(t+\tau) = E(t)E(\tau)$.
   (b) Show that for all $t \in \mathbb{R}$, $E(t)E(-t) = 1$.
   (c) Show that $E(t)$ is never 0.

3. $(**)$ This is similar to the previous exercise, but requires more work. This time, imagine that we know about existence and uniqueness of solutions for second order differential equations of the type

$$x''(t) + x(t) = 0, \quad x(0) = a, \ x'(0) = b,$$

but nothing about trigonometric functions. (For example from Theorem 1.5.5, we know that this equations has a unique solutions, and we can see this by introducing the state vector comprising x and x'.) We define the functions S and C as the unique solutions, respectively, to

$$S''(t) + S(t) = 0, \quad S(0) = 0, \ S'(0) = 1;$$
$$C''(t) + C(t) = 0, \quad C(0) = 1, \ C'(0) = 0.$$

(Privately, we know that $S(t) = \sin t$ and $C(t) = \cos t$.) Now show that for all $t \in \mathbb{R}$,

   (a) $S'(t) = C(t)$, $C'(t) = -S(t)$.
   (b) $(S(t))^2 + (C(t))^2 = 1$.
   Hint: What is the derivative?

(c) $S(t+\tau) = S(t)C(\tau) + C(t)S(\tau)$ and $C(t+\tau) = C(t)C(\tau) - S(t)S(\tau)$, all $\tau \in \mathbb{R}$.

(d) $S(-t) = -S(t)$, $C(-t) = C(t)$.

(e) There is a a number $\alpha > 0$ such that $C(\alpha) = 0$. (That is, $C(x)$ is not always positive. If we call the smallest such number $\pi/2$, we have a definition of π from differential equations.)