Extra Notes #1

Differential Equations

Theorem. *There are n real or complex eigenvalues of any multiplicity $\lambda_1, \ldots, \lambda_n$ for an $n \times n$ matrix A.*

Definition (multiplicity). The number of times an eigenvalue repeats in $\lambda_1, \ldots, \lambda_n$ for $A \in \mathbf{R}^{2\times 2}$ is its multiplicity.

Definition (conjugate). The conjugate of a complex number $c = \alpha + \beta i$ is given by $\bar{c} = \alpha - \beta i$ for $\alpha, \beta \in \mathbf{R}$ with $\beta \neq 0$. The conjugate can be thought of as a complex number's inverse.

Theorem. *Suppose an $n \times n$ matrix A contains only real entries. Then,*

- *(i) For every complex eigenvalue λ_i of A, the conjugate $\bar{\lambda}_i$ is also a eigenvalue of A.*
- *(ii) At least one eigenvalue λ_k is real if n is odd.*

Definition (Jordan-form classifications). All matrices of size 2×2 has a Jordan form that makes computing matrix eponentials significantly easier. The classification type depends on the form of the eigenvalues, which are shown below.

- *(i) For real eigenvalues $\lambda_1 > \lambda_2$, the Jordan form is given by* $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
- *(ii) For a double eigenvalue with two linearly independent eigenvectors, the Jordan form is the same as (i).*
- *(iii) For a double eigenvalue with only one eigenvector, the Jordan form is given by* $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$.
- *(iv) For a complex eigenvalue and its conjugate, the Jordan form is given by* $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

Definition (Jordan exponentials). The matrix exponential of the Jordan forms are given below and corresponds to the list above.

- *(i) For* $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$*, the matrix exponential is given by* $e^{tJ} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}$.
- *(ii) This case has the same form as (i) but with $\lambda_1 = \lambda_2$.*
- *(iii) For* $J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$*, the matrix exponential is given by* $e^{tJ} = \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix}$.
- *(iv) For* $J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$*, the matrix exponential is given by* $e^{tJ} = \begin{pmatrix} e^{t\alpha}\cos(t\beta) & e^{t\alpha}\sin(t\beta) \\ -e^{t\alpha}\sin(t\beta) & e^{t\alpha}\cos(t\beta) \end{pmatrix}$.

Definition (decoupled systems). Decoupling is the act of separating an n-dimensional system of differential equations into disjoint partitions with derivatives that only depend on their own partition.

> *Decoupled systems breaks down a larger system into subsystems that are independent of each other. Sometimes there are no distinct decouples. The full solution to the system is found by solving each subsystem partition and grouping all the solutions together. Decoupling can make larger systems easier to work with.*

See the actual notes for Jordan forms of 3×3, 4×4, and the general $n \times n$ form.